RECITATION 11

APPLICATIONS OF OPTIMIZATION

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Section 1. *Exercises*

Exercise 1

Cost of making $x > 0$ amount: $C(x) = x^2 + 5x - 8$.
Price of each: $p(x) = 9 - x$. Maximize profit.

Solution .:.

Profit is $P(x) = x \cdot p(x) - C(x) = (9x - x^2) - x^2 - 5x + 8 = -2x^2 + 4x + 8$. To maximize profit, $P'(x) = -4x + 4$ which is 0 iff $x = 1$. It's clear that this is a maximum by the second derivative test. Thus $P(1) = 10$ is the maximum profit.

Exercise 2

Cost of making $x > 0$: $C(x) = e^x$.
Minimize average cost.

Solution .:.

$A(x) = C(x)/x = e^x/x$. Using the product rule, $A'(x) = \frac{e^x}{x} - \frac{e^x}{x^2}$. $A'(x) = 0$ iff $e^x/x = e^x/x^2$ iff $x^2 = x$ iff $x = 1$ (recall that $x \neq 0$ for the average to make sense). $A''(x) = e^x/x - 2e^x/x^2 + 2e^x/x^3$ so that $A''(1) = e - 2e + 2e = e > 0$ and hence we have a minimum of $A(1) = e^1/1 = 1$.

Exercise 3

Cost of making $x > 0$: $C(x) = x^2 + 2x - 1$.
Price of each: $p(x) = \ln(x) + \sin(x)$.
Calculate marginal cost;
Calculate marginal revenue.

Solution .:.

Marginal cost is $C'(x) = 2x + 2$.
Marginal revenue is $\frac{\mathrm{d}}{\mathrm{d}x} x p(x) = \frac{\mathrm{d}}{\mathrm{d}x} x\ln(x) + x\sin(x)$. Using the product rule, this is just $\ln(x) + 1 + \sin(x) + x\cos(x)$.

Exercise 4

Cost of making $x > 0$: $C(x) = x^2 + 4$.
Calculate minimum average cost.

Solution .:.

Average cost is $A(x) = C(x)/x = x + 4/x$. To find the minimum of this function, we find critical numbers: $A'(x) = 1 - 4/x^2$ which is 0 iff $1 = 4/x^2$, i.e. $x = \pm 2$ and since $x > 0$, we get that $x = 2$. $A'(x) = (x^2 - 4)/x^2$ which has the same sign as $x^2 - 4$ (since dividing by a positive number like x^2 doesn't change the sign). For $0 < x < 2$, $A'(x) < 0$ since $x^2 < 4$. For $x > 2$, $A'(x) > 0$ since $x^2 - 4 > 0$. Thus this is a minimum for average cost, and this minimum average cost is $A(2) = 2 + 4/2 = 4$.

Exercise 5

Cost of making $x > 0$: $C(x) = x(10 - \cos(x))$.
Calculate minimum average cost.

Solution .:.

Average cost is $A(x) = C(x)/x = 10 - \cos(x)$. $A'(x) = \sin(x)$ which is 0 iff $x = \pi n$ for $n > 0$ an integer. At these values, $\cos(x) = \pm 1$ so that $A(2\pi) = A(4\pi) = \cdots = 10 - 1 = 9$ is the minimum average cost while $A(\pi) = A(3\pi) = \cdots = 10 - (-1) = 11$ is the maximum average cost.

Exercise 6

Cost of making $x > 0$: $C(x) = 2(x - 5)e^{x-5}$.
Price of each: $p(x) = e^{x-5}$.
Calculate the maximum profit.

Solution .:.

Profit is given by revenue $R(x) = x \cdot p(x)$ and cost: profit $P(x) = xp(x) - C(x) = xe^{x-5} - 2xe^{x-5} + 10e^{x-5} = 10e^{x-5} - xe^{x-5}$. To maximize $P(x)$, as always, we look at the critical points, and classify them. By the product rule,

$$P'(x) = 10e^{x-5} - e^{x-5} - xe^{x-5} = 9e^{x-5} - xe^{x-5} = (9 - x)e^{x-5}.$$

Since e^{x-5} is never 0, $P'(x) = 0$ iff $(9 - x) = 0$, i.e. $x = 9$. Now we can form a number line. For $x < 9$, $P'(x) > 0$ so that P is increasing there. For $x > 9$, $P'(x) < 0$ so that P is decreasing there. Hence by the first derivative test, $x = 9$ yields the max value of P: $P(9) = 10e^4 - 9e^4 = e^4$.